Spatial correlations:

Important physical quantity in science, especially phase transitions are spatial correlations. This means if something is true at place $x$, how likely is it to be true at place $x + \Delta x$. This was like $g(r)$, how much more likely are we to find a particle then any other given distance away from something else.

We also measured this as $\langle P(r) P(r') \rangle$ (or $\langle S_\text{p}(r) S_\text{p}(r') \rangle$).

In a lattice model, we compute the following related quantity:

$$C_{ij} = \langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$$

called the spin-spin correlation function.
Goes to zero when $\langle s_i; s_j \rangle \rightarrow \langle s_i \rangle \langle s_j \rangle$, which happens when uncorrelated, eg $|\langle s_i \rangle | \gg 0$, sometimes normalize

\[ C_{ij} \rightarrow \frac{\langle s_i; s_j \rangle - \langle s_i \rangle \langle s_j \rangle}{\langle s_i \rangle - \langle s_i \rangle^2} \]

This is connected to the susceptibility

\[ \chi = \frac{1}{N} \langle (S\cdot M)^2 \rangle \quad \text{where} \quad S\cdot M = \sum_{i=1}^{N} (s_i - \langle s_i \rangle) \]

\[ = \frac{1}{N} \sum_{i;j} \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \]

\[ = \frac{N}{N} \cdot \frac{1}{N} \sum_{i;j} \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle = \sum_{i;j} \frac{1}{2} C_{ij} \]
But we already saw that $X$ diverges in a phase transition, so this is a signature of ‘long range correlations’.

Two ways this can diverge:

1) 2 phase coexistence

Near the critical point, no distinction between two phases, so the divergence becomes large. This means correlations become long range.
usually \( G(r) = c_{ij} v e^{-r/\xi} \)

and as critical point is approached
\[ \xi \sim |T - T_c|^{-\nu} \]

The fact that this length scale gets big means the system looks the same on small & large length scales, which leads to the renormalization group idea!

Renormalization Group

"Coarse grain" over section of system try to have partition function look the same
Then \( \mathcal{H} = \sum_{\langle ij \rangle} J \mathbf{s}_i \mathbf{s}_j \rightarrow \sum_{\langle ij \rangle} J \mathbf{s}_i \mathbf{s}_j \)

Then you can repeat this procedure & iterate. If \( J \rightarrow J' \) process converges, then this is at a "fixed point" and it corresponds to a phase transition.

From the equations that generate \( J \rightarrow J' \), we can get the critical exponents too! We'll illustrate in 1d Ising model (no spontaneous mag transition) & discuss the result in 2d. [1 of different possible procedures]

@ zero field

\[ Q(k, N) = \sum_{s_1, s_2, \ldots, s_N} \exp \left[ k \left( s_1 s_2 + s_2 s_3 + \ldots + s_N s_1 \right) \right] \]

In mean field, averaged over all other
degrees of freedom but 1. Here, remove finite degrees of freedom.

Eg: Sum over even spins

\[
\begin{align*}
1 & \quad \frac{1}{2} & \quad \frac{3}{2} & \quad \frac{5}{2} & \quad \frac{7}{2} & \quad \frac{9}{2} \\
\Rightarrow & \quad 1 & \quad 3 & \quad 5 & \quad 7 & \quad 9
\end{align*}
\]

\[Q(k, N) = \sum_{s_1, s_3} \left( \exp[k(s_1 + s_3)] + \exp[-k(s_1 + s_3)] \right) \times \left( \exp[k(s_3 + s_5)] + \exp[-k(s_3 + s_5)] \right) \times \ldots \]

Want to express in same as original form.

If possible, but w/ new \(k'\) (new inverse temperature, or \(\beta\)).

Would be true if \(e^k(s + \tilde{s}') + e^{-k(s + \tilde{s}')} = f(k)e^{k's's'}\) for all \(s, s'\).

Then \(Q(k, N) = f(k)^{N/2} Q(k', N/2)\)

[Leads to transformation]
finding this is possible in 1d
\[ \text{in high order, have to approximate} \]
if \( s, s' \) in same direction
\[ e^{2k} + e^{-2k} = f(k) e^{2k'} \quad (1) \]
if opposite
\[ 2 = f(k) e^{-k'} \Rightarrow f(k) = 2 e^{k'} \quad (2) \]
we can solve for \( k' \) & \( f(k) \)

\[ (2) \Rightarrow (1) \]
\[ 2 e^{2k'} = e^{2k} + e^{-2k} \]
\[ \Rightarrow k' = \frac{1}{2} \log \left( \cosh(2k') \right) \]

plugging back in to 2
\[ f(k) = 2 \cosh^{1/2}(2k) \]

Consider \( \ln Q \) (a sort of free energy)
expect to grow \& \( N \). Define \( g(k) = \frac{1}{N} \ln Q \)
[extensive free energy]
\[ g(k) = \frac{1}{N} \left[ \frac{N}{2} \ln f(k) + \ln \delta(k', N/2) \right] \]
\[ N^{1/2} g(k') \]
\[ g(C') = 2g(k) - \ln f(k) \]
\[ = 2g(k) - \ln \left( 2 \sqrt{\cosh(2k)} \right) \]

If we know \( Q(k') \) for 1 value of \( k \), we can find it for other values.

In this renormalization, \( k' \) is always less than \( k \), \( \sinh(2k) = k' < k \).

Alternatively, swap \( k \) & \( k' \):

\[ k = \frac{1}{2} \cosh^{-1}(\exp(2k')) \]

& solve for \( g(k) \):

\[ g(k) = \frac{1}{2} g(k') + \frac{1}{2} \ln 2 + \frac{1}{2} \ln \left( \cosh(\cosh^{-1}(\exp(2k'))) \right) \]
\[ = \frac{1}{2} g(k') + \frac{1}{2} \ln 2 + k' \frac{1}{2} \]

What do we do with this? For small \( k' \), \( k' = 0.01 \), high temp or low \( \theta \) means spins close to uncorrelated, \( \Theta(k'/\theta) \approx 2^\theta \), \( g(k') = \ln 2 \approx 0.693 \).

\[ \Rightarrow \frac{k}{\theta} = 0.100534 \]
\[ g(k) = 0.698147 \]
Keep iterating & $k$ keeps growing
during the iteration, $g(k)$ gets closer & closer to the exact $g(k_c)$ for that
value of $k$ $C_\ln 1d$, can compute exactly

For no field & large $N$, we'd seen

$$Q(\beta_{1N}) = (2 \cosh(\beta_3))^n$$

$$= (e^k + e^{-k})^n$$

$$\ln Q/n = \ln (e^k + e^{-k}) = k + \ln (1 + e^{-2k})$$

for large $k_c$, $Q(k_c) \sim k$

Can start w/ eg $g(0) \times 10$ & iterate other equations to get smaller $k$

Results apparently grow in this procedure, but the picture is

\[
\begin{align*}
    & x \rightarrow \quad \rightarrow \quad \rightarrow \quad x \\
    k = 0 & \rightarrow \quad \rightarrow \quad \rightarrow \quad k = \infty \\
\end{align*}
\]

renormalization
Flow
for \( k=0 \) or \( k=10 \), params don't change "fixed point"

In 2d

\[
\begin{align*}
&x \leftarrow \leftarrow \leftarrow \leftarrow x \rightarrow \rightarrow \rightarrow \rightarrow x \\
&k=0 \\
&k=\kappa \\
&k=0
\end{align*}
\]

Unstable fixed point at

\( k_c \approx 0.50699 \) (for a similar problem

See Chaudhuri Ch 5)

where exact value \( J/k_bT_c = k_c = 0.44069 \)

See book pg 261 that shows it increases

needed as get closer & closer
to \( k_c \) grows, connecting to a
growing length scale as \( T_c \) approached