Lecture 25

Intro to Quantum Stat Mech, pt 2
Ideal Gas Distinguishable Particles (Neglect spin)

\[ Q(N, V, T) = \frac{g^n}{(\prod_{n} e^{-\beta\mu_n})^N} \]

\( u \) / Occupation #s,

\[ Q(N, V, T) = \sum_{\{f\}} g(\{f\}) e^{-\beta \sum_n m_n f_n} \]

\[ g(\{f\}) = \frac{N!}{\prod_{n} f_n!} \]

eg., 2 sh. 6. \( f = f_1 + f_2 \)

\[ g(f_1, f_2) = \frac{N!}{f_1! f_2!} = \frac{N!}{f_1! (N-f_1)!} \]

\[ Q(N, V, T) = \sum_{\{f\}} \frac{N!}{\prod_{n} f_n!} \prod_{n} e^{-\beta m_n f_n} = (\sum_{n} e^{-\beta m_n})^N \]

\[ \text{multinomial expansion} \]

So, 1 particle \( \sum_{n} e^{-\beta m_n} = \sum_{n} e^{-\frac{2\pi^2 m^2}{nh^2} \frac{1}{\text{ml}^2}} \)

when \( L \to \infty \), spacing becomes continuous, short

\[ \int dn e^{-\frac{2\pi^2 m^2}{nh^2} \frac{1}{\text{ml}^2}} = 4\pi \int_0^\infty dn e^{-\frac{2\pi^2 m^2}{nh^2} \frac{1}{\text{ml}^2}} \]

result

\[ V \left( \frac{m}{2\pi^2 p^2} \right)^{3/2} = \frac{1}{\sqrt{8}} \lambda^3 \]

case as classical ideal gas

All QM must be in spin multiplets
Identical fermions (bosons)

Excluding particles is same physical state, so

\[ g(\{f_i^p f_i^m\}) = 1, \text{ but} \]

\[ f_m^p = 0,1 \text{ for fermions} \]

\[ Q(\{N_i U_j T\}) = \sum_{f_{nm}} e^{-\beta E_m} f_{nm} e^{\beta E_n} = \sum_{f_{nm}} \prod_{i=1}^{N} e^{-\beta E_i} \]

\[ n \text{ restriction, } \sum_{n} f_{nm} = N \]

which is hard to keep track of

Instead better to use \( Z(\mu, U, T) = \frac{1}{2N} \int_0^\infty Q(C_N U, T) \)

\[ = \sum_{N=0}^{\infty} e^{\beta N} \sum_{f_{nm}} \prod_{i=1}^{N} e^{-\beta E_i} \]

\[ \text{Same results in } \lim \text{ still subject to occupation } f_1^p \text{ but outer sum reliever restriction as all things will be rated} \]

\[ e^{\mu} Z(\mu, U, T) = \sum_{f_{nm}} \prod_{i=1}^{N} e^{\beta E_i} \]

\[ \text{also all values of } N \text{ taken into account} \]

\[ B/c \sum \sum \sum \ldots e^{\beta (\mu - E_1)} f_1 e^{\beta (\mu - E_2)} f_2 \]

\[ = \left( \sum_{f_1} e^{\beta (\mu - E_1)} f_1 \sum_{f_2} e^{\beta (\mu - E_2)} f_2 \ldots \right) \]
\[ Z = \prod_m \prod_n \sum_{f_{nm}} e^{f_{nm}(\mu - \epsilon_n)} \]

for fermions \( f_{nm} = 0 \) or 1

\[ Z = \prod_m \prod_n \left( 1 + e^{f_{nm}(\mu - \epsilon_n)} \right) \quad \text{(fermions)} \]

For bosons, \( f_{nm} = 0 \to \infty \), like last line

\[ Z = \prod_m \prod_n \frac{1}{1 - e^{f_{nm}(\mu - \epsilon_n)}} \quad \text{(bosons)} \]

Product over \( m = 0 \ldots 2s+1 = g \)

\[ Z = \prod_n (1 + a e^{f_{nm}(\mu - \epsilon_n)})^{g \cdot \alpha} \]

\[ \alpha = 1 \quad \text{for fermions,} \quad -1 \quad \text{for bosons} \]

\[ \frac{P_0}{k_B T} = \ln Z = ga \sum_n \ln (1 + a e^{f_{nm}(\mu - \epsilon_n)}) \]

\[ = ga \sum_n \ln (1 + a e^{-\beta \epsilon_n}) \quad e^{\beta \mu}, \quad \text{fugacity} \]

\[ \langle N \rangle = \frac{\partial}{\partial a} \ln Z = \frac{a^2 \alpha}{g} \sum_n e^{-\beta \epsilon_n} \frac{1}{1 + a \epsilon e^{-\beta \epsilon_n}} = g \sum_n \frac{e^{\beta(\mu - \epsilon_n)}}{1 + a e^{\beta(\mu - \epsilon_n)}} \]
Now, comparing to a fixed $N$ situation

$$N = \sum_\alpha \sum_\nu f_{\nu \alpha}$$

$$\Rightarrow \langle N \rangle = \sum_\alpha \sum_\nu \langle f_{\nu \alpha} \rangle$$

$$\Rightarrow \text{we can associate } \langle f_{\nu \alpha} \rangle = \frac{e^{(\beta \mu - \epsilon)}}{1 + e^{\beta (\mu - \epsilon)}} \text{ for fermions}$$

$$\Rightarrow \frac{1}{1 + e^{-\beta (\mu - \epsilon)}} = \frac{1}{1 + e^{\beta (\epsilon - \mu)}} \quad \text{(fermi-drac distribution)}$$

Consider low temp, high $\beta$,

for $\epsilon_n - \mu > 0$, $\langle f \rangle \to 0$

$\epsilon_n - \mu < 0$, $\langle f \rangle \to 1$

$$\langle f_{\nu \alpha} \rangle = \sum_{\epsilon_n > \mu} 1 \quad \text{for } \epsilon_n < \mu = \Theta (\mu - \epsilon_n)$$

The energy where states above are unoccupied is the "fermi energy"
Ideal boson gas

Don’t have time to go into details, but boson gas can undergo Bose-Einstein condensation where at low T all particles go into the same state

\[ \langle n_{rm} \rangle = \frac{1}{e^{\beta (E_n - U)} - 1} \]

\[ \langle f_{rm} \rangle = \frac{e^{\beta n}}{1 - e^{\beta n}} \]

at some temp, can show
Black-body radiation

A hot, radiator object, will absorb & emit radiation, according to

\[ \frac{\alpha}{c^2} \]

\[ \propto \frac{\omega^2}{k_B T} \]

(The only good sum rule)

Photons have spin 1, bosons

also emitting & absorbing photons, so joust like our ground canonical treatment

Only starting waves commensurate w/ the size of the cavity are allowed, land of like particle in a box

Hence only \( k^2 = \frac{n^2}{L^2} \), \( n_x, n_y, n_z = 1, 2, ... \)

\[ E = \hbar c |k| = \hbar c k \]

\[ \text{momentum} = \hbar |k| = \hbar k \]

\[ k^2 = \frac{\pi^2}{L^2} \left( n_x^2 + n_y^2 + n_z^2 \right) \]

Number of waves w/ momentum \( k \) is volume of sphere \( \Phi(k) = \frac{4}{3} \pi |k|^3 / N = \frac{1}{6} \pi \left( \frac{|k|^3}{\pi} \right) = \frac{v}{b^2} / 6 \pi^2 \)

\( \text{average } |k| \text{d}k \)

\[ \omega(k) \text{d}k = \frac{\text{d}\Phi}{\text{d}k} \text{d}k = \frac{V k^2}{2 \pi^2} \text{d}k \]

\( \times 2 \text{ for } 2 \text{ polarizations} \)
One can show (no time here) that \( n = 0 \) for an ideal lens of photons.

In this case we know

\[
\langle E^2 \rangle = \frac{1}{e^{\beta E} - 1}
\]

\[
\Rightarrow \langle E^3 \rangle = \langle E^3 \rangle = \frac{3}{2} \frac{\hbar c k}{3} = \frac{3}{2} \frac{\hbar c k}{3} e^{\frac{\hbar c k}{e^{\beta E} - 1}}
\]

If \( k \)'s close together, can convert to integral w/ DOS:

\[
\langle E^3 \rangle \rightarrow \int dk \, w(k) \frac{\hbar c k}{2} e^{\frac{\hbar c k}{e^{\beta E} - 1}}
\]

\[
= \frac{\hbar c k}{\pi^2} \int_0^\infty dk \, \frac{k^3}{e^{\frac{\hbar c k}{e^{\beta E} - 1}}}
\]

\[
= \frac{\hbar c k}{\pi^2} \int_0^\infty dw \, \frac{w^3}{e^{\frac{\hbar c k}{e^{\beta E} - 1}}}
\]
\[
\frac{3}{\pi} = \int_0^\infty P(w,T) \, dw
\]

\[
\Rightarrow P(w,T) = \frac{k}{\pi^2 c^3} \frac{w^3}{e^{\frac{w}{kT}} - 1}
\]

@ low w, \[ e^{\frac{w}{kT}} \approx 1 + \frac{w}{kT} \]

\[
\Rightarrow P(t_\nu < c^3 kT, T) \propto \frac{t_\nu}{\pi^2 c^3} \frac{w^3}{e^{\frac{w}{kT}}}
\]

\[
\propto \frac{w^2 kT}{c^3} \frac{1}{e^{\frac{w}{kT}}}
\]

@ high freq \[ P(w,T) \propto \frac{w^3}{e^{\frac{w}{kT}}} \]

Summary: in stat mech, one consider averages over many particles & it gives us powerful tools & can derive sim methods to solve problems where exact solutions seem impossible!
We also correct microscopic quantities to bulk observable measurements.

- But more importantly, can predict fluxes in small # experiments & heterogeneous environments.
- And may be even out of eq.

- Hopefully, you can apply these ideas in your own work!