Phase transitions: Part 2
Last time: discussed general physics of phase transitions

We introduced the Ising Model as a simple way to study magnetization hase transition

\[ H = - \frac{1}{2} \sum_{i,j} J s_i s_j - h \sum_i s_i \] (in dimensions)

& PBC

Q: When is there a spontaneous magnetization transition? \[ m = \frac{M}{N} = \frac{\langle \sum s_i \rangle}{N} = \langle s \rangle > 0, v h = 0 \]

Said, should not exist for 1d ising model

For 1d model, \( h=0 \), said

\[ Z \propto (2 \cosh(\beta J))^N \]

and \[ f = \frac{\langle s \rangle}{N} = -kT \frac{\partial \ln Z}{\partial T} = -\frac{1}{kT} \log (2 \cosh (\beta J)) \]

Not: \( h \neq 0 \)

Useful model for binding

But what about \( h \neq 0 \)?
Need a new technique called transfer matrices.

\[ P_{s,s'} \] is the matrix w/ entries \( e^{\psi(s,s') + \theta h(s,s')} \)

So \[ \langle 1|P|1 \rangle = e^{\psi} \]

\[ \langle 1|P|0 \rangle = \langle 0|P|1 \rangle = e^{-\psi} \]

\[ P = \begin{pmatrix}
  e^{\psi} & e^{-\psi} \\
  e^{\psi} & e^{-(\psi)}
\end{pmatrix} \]

So \[ Z = \sum_{s_1, \ldots, s_N} \langle s_1 | P^n | s_1 \rangle \]

Spin eigenvectors are complete \( \implies Z = \sum_{s_1} \langle s_1 | P^n | s_1 \rangle = \text{Tr}(P^n) \)

\[ \text{Tr}(M^n) = \text{Tr}(U D^n U^*) = \sum_{i=1}^{N} \lambda_i^n \]

are eigenvalues of \( M \) if \( M \) diagonalizable.

\[ Z = \lambda_1^n + \lambda_2^n \]

\[ \text{Det}[P - \lambda I] = 0 \]

\[ \lambda = e^{\psi} \left( \cos(\theta h) \pm \sqrt{\sin^2(\theta h) + e^{-4\psi}} \right) \]

as \( N \to 0 \)

\[ Z \propto \lambda^n \]
\[
\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}
\]

and
\[
I(h, \beta) = -\frac{1}{\beta} \log \left[ e^\beta \cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-2\beta h}} \right]
\]

\[
M(h, \beta) = \frac{\partial I}{\partial \beta} = \frac{\sinh(\beta h) + \frac{1}{2} (\sinh^2(\beta h) + e^{-4\beta h})^{-\frac{1}{2}} \cosh(\beta h)}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta h}}}
\]

as \( h \to 0, \sinh(\beta h) = 0, \cosh(\beta h) \to 1 \)

So \( \lim_{h \to 0} M(h, \beta) = 0 \)

@ \( \beta < \infty \),

\[ \text{if } \beta \to \infty (T \to 0), \quad e^{-4\beta h} \to 0 \quad \text{and} \]

\[
m \to \frac{\sinh(\beta h) \pm \cosh(\beta h)}{\cosh(\beta h) \pm \sinh(\beta h)} \to \pm 1
\]

and which depends on \( h > 0 \) or \( h < 0 \)

so phase transition / critical point at \( T \to 0, \) the \( h > 0 \)

\[
\begin{array}{ccc}
\uparrow & \quad \downarrow & \quad \uparrow \\
\text{sup} & \quad \text{inf} & \quad \text{sup}
\end{array}
\]

What about higher dimensions? We can solve approx using a technique called "mean-field theory", meaning each spin feels the avg effect of its neighbors

let's call \( S_{\downarrow} = s_i - m \Rightarrow s_i = S_{\downarrow} + m \)
Then \( H = -\frac{1}{2} \sum_{i<j} s_i s_j - \frac{1}{2} \sum_i s_i \langle \phi \rangle \)

\[ = -\frac{1}{2} \sum_{i<j} (m^2 + s_is_j)(m + s_is_j) - \frac{1}{2} \sum_i s_i \langle \phi \rangle \]

\[ = -\frac{1}{2} \sum_{i<j} (m^2 + m(s_i-m) + h(s_i-m)) - \frac{1}{2} \sum_i s_i \langle \phi \rangle \]

\[ = \sum_i (m^2) - \sum_{i<j} (s_i + s_j) - \frac{1}{2} \sum_i s_i \langle \phi \rangle \]

\[ = \sum_i m^2 - (h+2m \beta) \sum_i s_i \]

So \( Z = \sum_{s_1, s_2 \ldots s_N} e^{-\beta \sum m^2 s_i} + \beta (h + 2m \beta) \sum s_i \]

\[ = e^{-\beta \sum m^2 s_i} \left[ 2 \cosh \left( (h + 2m \beta) \right) \right]^N \]

\[ \langle m \rangle = \frac{1}{N} \frac{\partial \log Z}{\partial \beta} = \frac{\partial \log \left[ 2 \cosh \left( (h + 2m \beta) \right) \right]}{\partial \beta} = \frac{\sinh (h + 2m \beta)}{\cosh (h + 2m \beta)} = \tanh (h + 2m \beta) \]

No analytical solution, can get numerically

@ \( h = 0 \), spontaneous mag?
For low $T$, 3 solutions:
$\bar{m} = 0$, trivial, not as interesting
Other solution where lines cross

$2B\bar{J}z = 1$ separates the regimes, & hence

$k_B T_c = 2Jz$ gives predicted critical point

For $1/2$, wrong, no $T_c > 0$